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On the Lack of Exact Controllability for Mild Solutions in Banach Spaces

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It is shown that exact controllability in finite time for linear control systems given on an infinite dimensional separable Banach space in integral form (mild solution) can never arise using locally L_1-controls, if the operator through which the control acts on the system is compact. This improves a previous result of the author, by removing the assumption that the state space have a basis. It is suggested by the recent discovery that a separable Banach space need not have a basis.

1. INTRODUCTION, BACKGROUND AND STATEMENT OF MAIN RESULT

Consider the control process described by the following integral model

$$x(t, x_0, u) = S(t)\, x_0 + \int_0^t S(t-\tau)\, Bu(\tau)\, d\tau, \qquad t \geqslant 0, \tag{1.1}$$

under the following standard assumptions: $x(t, \cdot, \cdot)$ belongs to a separable Banach space X (state space); $u(t)$ is a U-valued function, locally L_1 (control function), where U (control space) is also a Banach space; $S(t)$, $t \geqslant 0$, is a strongly continuous semigroup of bounded operators (of class C_0); B is a bounded operator: $U \to X$; finally $x_0 \in X$. The integral is well defined in the sense of Bochner. Unless otherwise stated, X will be assumed infinite dimensional. Also, (1.1) is (strongly) continuous in t [4, pp. 88]. See [2–4, 6, 7] for the necessary background for vector valued functions. The choice of the above model is justified by the following considerations.

(i) If $S(t)$ is, in particular, uniformly continuous—which happens just in case its infinitesimal generator A is bounded on X [3, pp. 621]—(1.1) is the unique (a.e.) solution of the abstract differential equation:

$$\dot{x} = Ax + Bu, \qquad \text{with} \qquad x(0) = x_0 \in X.$$

(ii) If $S(t)$ is, instead, only strongly continuous with unbounded infinitesimal generator A, and $u(t)$ is suitably smooth (e.g. C^1), (1.1) is still the unique solution of: $\dot{x} = Ax + Bu$, at least when $x_0 \in D(A)$, the domain of A, dense in X [5, pp. 486], [7, pp. 30]. Other conditions under which (1.1) is the (perhaps a.e.) unique solution of the correspondent differential equation are given in [1, pp. 154] [5, pp. 491]. It is customary to refer to (1.1), for a locally L_1 function $u(t)$, as "mild solution" of the correspondent differential equation.

Both the norm of X and the norm of U will be denoted by $\| \, \|$.

We say that (1.1) is approximately controllable on $[0, T]$, $0 < T < \infty$ (in finite time) in case: given x_0 and x_1 in X and $\epsilon > 0$, there is a $L_1[[0, T], U]$ — control u (a $L_1 [[0, t_1], U]$ — control u for some time t_1, possibly depending on ϵ, x_0, x_1) such that

$$\| x(T, x_0, u) - x_1 \| \leqslant \epsilon, \qquad (\| x(t_1, x_0, u) - x_1 \| \leqslant \epsilon).$$

We say that (1.1) is exactly controllable on $[0, T]$ (in finite time) in case $\epsilon = 0$ in the previous definitions. In case (ii) above, if one instead takes the differential equation as primary model, the problem of its exact controllability in finite time can be quickly dismissed. In fact the solution $x(T, x_0, u)$ of the differential equation at the time T always lies in $D(A)$, which is only dense in X [9, p. 166]. In other words, in case (ii), the correspondent differential equation cannot be exactly controllable in finite time.

In the present paper, we shall investigate the problem of exact controllability in finite time for the mild solution (1.1).

Recently the author has shown that[1]

THEOREM 1.1 [11, Section 3.3]. *Let X be infinite dimensional. Then, the system* (1.1) *is never exactly controllable in finite time, using locally L_1-controls, if:*

(i) *either B is of finite dimensional range*

(ii) *or X has a Schauder basis and B is compact.*

The proof was actually given for the case when A is bounded, but it was remarked in Remark 3.3.2 of [11] that the same proof carries over when A is simply the infinitesimal generator of a strongly continuous semigroup. Examples also were given in [11] of approximately controllable systems on an arbitrary finite interval $[0, T]$ that yet are not exactly controllable in finite time. If the assumption on B is relaxed, (1.1) may indeed be exactly controllable on $[0, T]$ (e.g., $X = U$, $A = 0$, $B = I$ (identity); for further examples and considerations, see the last part of Section 3.3 in [11]. Also

[1] See also [14] for a different proof of part (i) of Theorem 1.1 with respect to the class of controls locally of bounded variation.

recently, the long-standing open problem on whether every separable Banach space has a Schauder basis [8, 10] has been solved in the negative sense [8, footnote p. 878] by P. Enflo [12]; see also [13].

It is therefore desirable to remove the assumption that X has a basis from Theorem 1.1. Although bases have actually been constructed for all the known separable Banach spaces of physical interest [10], one may wonder for instance whether a theorem like Theorem 1.1 still holds for their infinite dimensional subspaces. The purpose of this note is therefore to remove the assumption that X have a basis from Theorem 1.1(ii) above and to present the following improvement.

THEOREM 1.2. *Let X be infinite dimensional. Then the system* (1.1) *is never exactly controllable in finite time using locally L_1-controls, if B is compact.*

The proof of the theorem will be given in the next section. We recall, that in [11], the assumption that X have a basis was used to approximate the compact operator B by a sequence of operators B_k of finite dimensional range, converging to B in the uniform operator topology: this then allows one to use part (i) of Theorem 1.1. Before giving the proof, some notation is in order. A vector $\tilde{u}$ of $L_p[[0, T], U]$, $1 \leqslant p \leqslant \infty$, is a U-valued function $u(t)$, $0 \leqslant t \leqslant T$ (technically, an equivalence class of functions differing on a set of measure zero [4, p. 82]), with standard norms [4, p. 89]. The unit sphere in U, $L_1[[0, T], U]$ and $L_\infty[[0, T], U]$ will be denoted, respectively, by U_1, $\tilde{U}_1$ and $\tilde{U}_1{}^\infty$. The strong closure of a set will be indicated by a bar, occasionally by Cl for simplicity of printing. Notice that the theorem is obvious if e.g., B commutes with A [5, p. 171], hence with $R(\cdot, A)$ [5, p. 173], finally with $S(t)$: use [3, Theorem 11, p. 622] and uniqueness of the Laplace transform.

2. PROOF

The proof is based on two Steps.

Step 1. An essetial point of the proof consists in showing that the operator Q, defined by

$$Q\tilde{u} = \int_0^T S(T-t)\, Bu(t)\, dt,$$

from $L_1[[0, T], U]$ into X is compact. (Linearity and boundedness of Q are trivial).

Step 2. Exactly the same arguments as in [11, Section 3.3] based on category arguments will finish the proof and hence will not be repeated. We therefore confine ourselves to show the result expressed in Step 1 above. We start with a lemma.

LEMMA 2.1. *Let the operator B be compact. Then the set*

$$\cup \{S(T-t)\,Bu(t);\,0 \leqslant t \leqslant T\} = M,$$

is precompact in X; here the union is taken over all measurable control functions $u(t)$ such that $u(t) \in U_1$, the unit sphere in U, $0 \leqslant t \leqslant T$.

Proof. First, notice that the following set inclusion holds

$$M \subset \{S(T-t)\,y\colon 0 \leqslant t \leqslant T;\, y \in \overline{BU_1}\} = K$$

and so it suffices to show that K is compact in X. To do that, observe that the function: $S(T-t)\,y$, from $[0, T] \times \overline{BU_1} \to X$ is continuous (in both arguments), as it follows from

$$\begin{aligned}
&\| S(T-t)\,y - S(T-t_0)\,y_0 \| \\
&\quad \leqslant \| S(T-t)\,y - S(T-t)\,y_0 \| + \| S(T-t)\,y_0 - S(T-t_0)\,y_0 \| \\
&\quad \leqslant k\,e^{\alpha T} \| y - y_0 \| + \| S(T-t)\,y_0 - S(T-t_0)\,y_0 \|
\end{aligned}$$

and the strong continuity of $S(t)$.

(Above, $k > 0$ and α is any constant greater than:

$$\omega_0 = \lim_{t\to\infty} \ln \| S(t) \| / t < \infty$$

[2–4, 7]).

Now, if B is compact, the set $[0, T] \times \overline{BU_1}$ is compact. Hence the set K is also compact, being the continuous image of a compact set [9, p. 158]. Q.E.D.

Since admissible controls on $[0, T]$ are traditionally taken to be $L_\infty[[0, T], U]$-functions, we wish to explicitly deduce, as a direct corollary of the previous lemma, the following particular case of Theorem 1.2, which will be used in the sequel.

THEOREM 2.2. *Let X be infinite dimensional. Then the system* (1.1) *is never exactly controllable in finite time, using locally L_∞-controls, if B is compact.*

Proof. We first show that the operator

$$\underset{\sim}{Q}\tilde{u} = \int_0^T S(T-t)\,Bu(t)\,dt,$$

this time from $L_\infty[[0, T], U]$ into X is compact. Consider the unit sphere of $L_\infty[[0, T], U]$:

$$\tilde{U}_1^\infty = \{\tilde{u}\colon \|| \tilde{u} \||_\infty = \text{ess sup} \| u(t) \| \leqslant 1\}.$$

Then each vector $\tilde{u}$ in $\tilde{U}_1^\infty$ is a U-valued function $u(t)$ contained in the unit sphere U_1 of U for almost all t in $[0, T]$. Modification of $u(t)$ on a set of measure zero of $[0, T]$ as to have $u(t) \in U_1$ for all t in $[0, T]$ does not effect the integral on the right side defining Q and hence the image set of $\tilde{U}_1^\infty$ under Q. Apply then the previous lemma to get that the set $\overline{M}$ in X there defined is compact. The very definition of the integral defining Q gives:

$$Q\tilde{U}_1^\infty \subset T\,\overline{\mathrm{co}}(\overline{M}),$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull. By Mazur's theorem [3, pp. 416], the set $\overline{\mathrm{co}}(\overline{M})$ is compact in X. Hence the operator Q, from $L_\infty[[0, T]; U]$ into X, is compact. Q.E.D.

We are ready now to present the full proof of the content of the first step for Theorem 1.2.

Remark 2.1. First of all notice that the argument given above for $L_\infty[[0, T], U]$ controls is not applicable when working with $L_1[[0, T], U]$ controls. This is so since the U-valued function $u(t)$ corresponding to a vector $\tilde{u}$ in the unit sphere $\tilde{U}_1^1$ of $L_1[[0, T], U]$

$$\tilde{U}_1^1 = \left\{\tilde{u}: ||| \tilde{u} |||_1 = \int_0^T \| u(t)\| \, dt \leqslant 1\right\},$$

may very well lie outside U_1 over a subset of $[0, T]$ of positive measure. Actually, one easily sees that

$$\cup \{Bu(t), 0 \leqslant t \leqslant T\} = BU,$$

where the union is taken over all $\tilde{u} \in \tilde{U}_1^1$. Moreover, it is not hard to realise that the set

$$\{S(T-t)\,BU, 0 \leqslant t \leqslant T\},$$

"naturally" corresponding to the set M of Lemma 2.1 may very well be a subspace (e.g., when $X = U$ and A and B commute, with A bounded). So an alternate route will be devised.

Proof of Step 1. 1. Consider the X-valued function $v_u(t)$, depending on the control $u(\tau)$, $0 \leqslant \tau \leqslant t$, defined by

$$v_u(t) = \int_0^t S(t-\tau)\,B\mu(\tau)\,d\tau$$

with

$$\mu(\tau) = \int_0^\tau u(s)\,ds, \qquad 0 \leqslant \tau \leqslant t.$$

If $u(\tau) \in \tilde{U}_1^1$, $0 \leqslant \tau \leqslant t$, then $\mu(\tau) \in \tilde{U}_1^\infty$, $0 \leqslant \tau \leqslant t$. By Theorem 2.2, the set

$$K_t = \left\{ v_u(t), \text{ for all } u(\tau), \text{ such that } \int_0^t \| u(\tau) \| \, d\tau \leqslant 1 \right\}$$

(set of attainability from the origin corresponding to controls $\mu(t)$) is pre-compact; here the subindex t denotes the dependence on t.

2. Next, notice that

$$\begin{aligned} \frac{d}{dt} v_u(t) &= \frac{d}{dt} \int_0^t S(\tau) \, B \left(\int_0^{t-\tau} u(s) \, ds \right) d\tau \\ &= \int_0^t S(\tau) \, Bu(t-\tau) \, d\tau = \int_0^t S(t-\tau) \, Bu(\tau) \, d\tau, \end{aligned}$$

see [7, pp. 31], for the detailed computation, extending the classical formula for differentiating an integral depending on a parameter, to our present X-valued case[2] see also [5, pp. 487].

In particular,

$$\left. \frac{dv_u(t)}{dt} \right|_{t=T} = \int_0^T S(T-t) \, Bu(t) \, dt = x(T, 0, u).$$

3. Also, it is plain that the compact sets $\bar{K}_t$ defined above are increasing in the sense that

$$0 \leqslant t' \leqslant t'' \qquad \text{implies} \qquad \bar{K}_{t'} \subset \bar{K}_{t''} \, .$$

In fact, a point reachable from the origin over $[0, t']$ using a control $u'(\tau)$, $0 \leqslant \tau \leqslant t'$, with

$$\int_0^{t'} \| u'(\tau) \| \, d\tau \leqslant 1$$

is also reachable over the longer interval $[0, t'']$ by applying the control

$$u''(t) = \begin{cases} 0 & 0 \leqslant t \leqslant t'' - t' \\ u'(t - (t'' - t')) & t'' - t' < t \leqslant t'' \end{cases}$$

where

$$\int_0^{t''} \| u''(t) \| \, dt \leqslant 1.$$

[2] Actually the computation in [7, p. 31] is applicable to a continous $u(t)$ and is carried out for a Riemann-type of integral; the same holds in the a.e. sense for a Lebesgue-type of integral; see [4, p. 88 after Corollary 2]. Alternatively, one can restrict in advance to continuous controls $u(\tau)$ in $\tilde{U}_1^1$, $0 \leqslant \tau \leqslant t$, and use the first part of Remark 2.2; the point is showing that the set inclusion (2.2) in Remark 2.2 holds.

Hence

$$\bigcup_{0 \leqslant t \leqslant T+1} \bar{K}_t = \bar{K}_{T+1}\,.$$

In other words, the totality of trajectories $v_u(t)$, $0 \leqslant t \leqslant T+1$, corresponding to all controls $u(\tau)$, $0 \leqslant \tau \leqslant t$, in $\tilde{U}_1{}^1$ lie in the compact set $\bar{K}_{T+1}$.

4. Denoting by $(\bar{K}_{T+1} + (-\bar{K}_{T+1}))$ the direct sum of $\bar{K}_{T+1}$ and $-\bar{K}_{T+1}$, we now want to show the following set inclusion

$$Q\tilde{U}_1{}^1 = \left\{\frac{d}{dt}\,[v_u(t)]_{t=T} = x(T, 0, u) \text{ for all } u(\tau),\, 0 \leqslant \tau \leqslant t, \text{ in } \tilde{U}_1{}^1\right\}$$

$$\subset k\, e^{\alpha(T+1)}\, \|\, B\, \|\, \overline{\mathrm{co}}(\bar{K}_{T+1} + (-\bar{K}_{T+1})) = C,$$

where k and α are as in the proof of Lemma 2.1. This will finish the proof of Step 1, since again by Mazur's theorem [3, pp. 416] the set C is compact.

In fact, for each h real, with $|\, h \,| < \min(T, 1)$, the vector $v_u(T+h) - v_u(T)$ obviously belongs to $\overline{\mathrm{co}}(\bar{K}_{T+1} + (-\bar{K}_{T+1}))$. On the other hand, the limit, as $h \to 0$, of $[v_u(T+h) - v_u(T)]/h$ exists and is in fact the vector

$$x(T, 0, u) = \int_0^T S(T - \tau)\, Bu(\tau)\, d\tau,$$

whose norm is less than $k\, e^{\alpha(T+1)}\, \|\, B \|$ for all $u(t)$, $0 \leqslant t \leqslant T$, in $\tilde{U}_1{}^1$. Therefore the vector $[v_u(T+h) - v_u(T)]/h$ belongs to C for all h suitably small. Since C is compact, in particular closed [9, pp. 158], the limit also belongs to C and the proof of Step 1 is complete. Q.E.D.

Remark 2.2. In the case when A is bounded on X, the following, perhaps simpler proof of Step 1 can be given. In this case,

$$S(t) = e^{At}, \qquad -\infty < t < \infty.$$

First notice that given $\tilde{u}$ in the unit sphere $\tilde{U}_1{}^1$ of $L_1[[0, T], U]$, there is a sequence of $\tilde{u}_n$, whose corresponding functions $u_n(t)$ are continuous on $[0, T]$, such that $\tilde{u}_n \to \tilde{u}$ [4, p. 86], i.e.,

$$\int_0^T \|\, u_n(t) - u(t) \|\, dt \to 0.$$

Moreover, we may require that $|||\, \tilde{u}_n \,|||_1 \leqslant 1$, $n = 1, 2, \ldots$.

Denoting by $\tilde{U}_{1c}$ the totality of vectors $\tilde{u}$ of $\tilde{U}_1{}^1$ whose corresponding functions $u(t)$ are continuous on $[0, T]$, the above says that: $\mathrm{Cl}\, \tilde{U}_1^c = \tilde{U}_1^1$, where Cl denotes (strong) closure. Since the operator Q

$$Q\tilde{u} = \int_0^T S(T-t)\, Bu(t)\, dt$$

from $L_1[[0, T], U]$ into X is bounded, we have

$$\mathrm{Cl}\,(Q\tilde{U}_{1c}) = \mathrm{Cl}\,(Q[\mathrm{Cl}\,\tilde{U}_{1c}]) = \mathrm{Cl}\, Q\tilde{U}_1^1.$$

Therefore it remains to show that

$$\mathrm{Cl}\, Q\tilde{U}_{1c} \subset \text{compact set.} \tag{2.2}$$

Let now $u(t)$ be continuous on $[0, T]$ and $\int_0^T \| u(t)\|\, dt \leqslant 1$. Then, the X-valued function

$$e^{A(T-t)}B \int_0^t u(s)\, ds, \qquad 0 \leqslant t \leqslant T,$$

is differentiable in t with continuous derivative equal to

$$\frac{d}{dt}\left[e^{A(T-t)}B \int_0^t u(s)\, ds\right]$$
$$= -\, e^{A(T-t)}AB \int_0^t u(s)\, ds + e^{A(T-t)}Bu(t), \qquad 0 \leqslant t \leqslant T.$$

Hence [7, Theorem 1.3.3 and 1.3.4, p. 6]

$$\int_0^T e^{A(T-t)}Bu(t)\, dt = \left[e^{A(T-t)}B \int_0^t u(s)\, ds\right]_0^T + \int_0^T e^{A(T-t)}AB \left[\int_0^t u(s)\, ds\right] dt$$
$$= B\int_0^T u(t)\, dt + \int_0^T e^{A(T-t)}AB\mu(t)\, dt,$$

with

$$\mu(t) = \int_0^t u(s)\, ds \qquad \text{and} \qquad \| \mu(t)\| \leqslant 1, \qquad 0 \leqslant t \leqslant T.$$

The above is nothing but the integration by parts formula for abstract functions. Since AB is also compact, Theorem 2.2 applies. Hence the set

$$K_1 = \left\{\int_0^T e^{A(T-t)}AB\mu(t)\, dt, \text{ for all } u(t) \text{ continuous in } [0, T]\right\},$$
$$\text{with } \||\tilde{u}\||_1 \leqslant 1,$$

is pre-compact. We then can write

$$\mathrm{Cl}\,Q\tilde{U}_{1c} \subset \bar{K}_1 + \overline{BU}_1$$

and the proof is complete, since the direct sum of two compact sets $\bar{K}_1$ and $\overline{BU_1}$ is also compact. Q.E.D.

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